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The Shadow Economy and Public Policy: a Real Option Analysis

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Abstract

In this paper I apply real option theory, in the aim to model tax compliance, enforcement and the shaping of a shadow economy in a general framework of mutual relationships, expectations and bargaining between a public authority and private agents. As an element of a social contract, tax obligations are considered a form of liability options, a concept only recently developed in the economic literature. By expanding this concept to the structural aspects of the informal sector, the paper develops a model to explain the shadow economy as a byproduct of non compliance, contractual features and tax design. Applications are also presented for the specific case of the Italian economy and the estimates of the size of its informal sector.

Keywords: Shadow economy, tax reform, noncompliance, decentralized enforcement.
JEL: D62, D81, H26, C78

The Shadow Economy and Public Policy: a Real Option Analysis

1. Taxation, public interest and real options

The extension of the shadow economy and tax non-compliance and its exponential increase over the last decades in both developed and developing economies is a phenomenon that challenges economic research and policy prescriptions. For example, in a study on a sample of 117 countries for 20 years, Schneider and Kepler (2016) estimate a range for the shadow economy going from a minimum value of 5.8 % for Switzerland to a maximum of 46 % for Bolivia, with Italy sitting in the middle with an average of 18% (rank 49). While these figures may appear very high (the current estimates for Italy revolve around 15% of the economy[1]), recent evidence on these orders of magnitude is presented in Goel and Nelson (2016) and in Markellos, Psychoyios and Schneider (2016) both on different measures and the direct and indirect effects of the shadow economy for a large sample of developed and developing countries. The basis of the shadow economy lies in the heterogeneous degree of tax compliance and enforcement that we find in most countries. These two phenomena appear inextricably linked as tax authorities in various countries try to respond to the persistence and widening of the shadow economy with harder sanctions and more effective

[1] The official estimate by ISTAT for the shadow economy and tax evasion in Italy is 13% of GDP in 2016. Other sources, however, obtain much larger figures from different databases (e.g. Eurispes estimates 540 Billion of combined evasion and illegal activities, which is almost 30% of 2016 GDP. (https://www.panorama.it/economia/tasse/quanto-vale-leconomia-in-nero-in-italia/)

methods of monitoring, control and prosecution of non compliant tax payers. In the crime and punishment economic approach (Becker, 1968), harsher fines and more effective enforcement are assumed to be the key to combat tax evasion (Allingham & Sandmo, 1972; Srinivasan, 1973). This implies that taxpayers behave as rational agents, and that their degree of tax compliance depends on the expected penalty and the risk of being caught. However, as Alm, Sanchez, & De Juan (1995, p. 15) aptly note: ”… a government compliance strategy based only on detection and punishment may well be a reasonable starting point but not a good ending point. Instead what is needed is a multi-faceted approach that emphasizes enforcement, as well as such things as positive rewards from greater tax compliance, the wise use of taxpayer dollars, and the social obligation of paying one's taxes. (Alm, Sanchez, & De Juan, 1995, p. 15)”.

In line with this suggestion, in this paper I apply real option theory, in the aim to model tax compliance and enforcement in a more general framework of mutual relationships, expectations and bargaining between a public authority and private agents. As an element of a social contract, in fact, tax obligations can be seen as an application of the notion of liability options, a concept only recently developed in the economic literature (Scandizzo and Knudsen, 2010; Scandizzo and Ventura, 2006, 2010, 2015, 2016). This paper follows this literature and applies to tax design theory several results already obtained in different contexts, which offer both extension and innovation opportunities to re-interpret some of the traditional tenets of public finance.

An option is defined as the faculty (but not the obligation) to purchase (in the case of the call option) or sell (for the put option) an underlying asset at a given price. In the case of financial calls and puts, it is evident that the faculty given by the option to its holder corresponds to an obligation for the party who has issued the option. In the case of a call option, in particular, the issuer will be forced to sell at the price agreed (the “strike”) if either at expiration or at the time of exercise such a price is below the market price. The call option is thus a contingent asset (i.e. an asset whose value is contingent upon the market price of the underlying) for its holder, while it is a contingent liability for its issuer. Similar considerations apply, mutatis mutandis, to the financial put option.

In the case of real options, the principle that the creation of a faculty always corresponds to the creation of an obligation can also be asserted, but not without several complications. First, a real option constitutes a faculty and not necessarily a right, in the twofold sense, that it is not necessarily enforceable and may not even be recognized by law. Thus, for example, one may have the option to choose a career or a business activity, but, aside from a general recognition of the right not to be interfered with in the pursuit of one’s interests, the corresponding faculty is not enforceable through the judicial system.

Second, even though one might be entitled in principle to some action, under the implicit or explicit protection of the law, the entitlement will be in practice ineffective in the absence of both the capability and of concrete opportunities to exercise the faculty in question. For example, a person may be entitled to reside in any neighborhood of a given city. Such a faculty is empty, however, if she does not have the money (the capability) to do so, or, even though she does, no house for sale ever comes up in the neighborhood of her choice.

Third, if the three elements of an effective entitlement are present (i.e. faculty, capability and opportunity), we can say that we face a real option. To such an option, however, contrary to the case of financial options, no obligations from third parties will necessarily correspond. This is because real options are often non contractual in nature, or are the result of the exercise of general liberties that can be at most ascribed to a general social contract, where the counterpart is society at large. The absence of specific obligations, however, does not imply that faculties can be created freely, but rather that the contingent liabilities generated by the assignment of contingent assets are more subtle and widespread than in the case of specific covenants. This is the case, in particular, of tax obligations, since they arise as the consequence of implicit contracts, sanctioned by law, that require residents of a given state (or other government domain) to pay a given amount in response to the options (i.e. a collections of rights without specific obligations to exercise them) offered by the resident status. The fact that tax obligations are sanctioned by law, on the other hand, gives the tax authority the option to recover the payment of the taxes in the case of non-compliance on the part of the resident. The option to evade the tax thus represents an opportunity in terms of contingent wealth for the resident, while it is a contingent liability for the tax authority. Conversely, the option to recover the tax through direct actions supported by law creates the possibility of a litigation between the public and the private party, where the public owns the contingent wealth of possible tax recovery and the private agent bears the contingent liability of the payment of the taxes and the related sanctions.

In order to characterize the obligations created by real options in the context of a social contract, we must start from the concept of freedom. A popular, if elementary way to define this concept is in terms of lack from interference: in the absence of specific provisions, each citizen has the right, in other words, to carry out all actions that she deems appropriate, unless these actions interfere, with citizens' freedom and, by threatening a similar right, on the actions that can be carried out by somebody else. This very general definition creates potentially two classes: (i) the actions that do not interfere with the other citizens' freedom, and (ii) the actions that do. The faculty to perform class (i) actions creates, if both capability and opportunities are present, real options of the contingent asset variety. The faculty to perform class (ii) actions, on the other hand, creates real options of both contingent asset and liability variety. Actions that infringe in other people's freedom, in fact, can still be performed, but are under the threat of counteractions from these people or from society at large (through the legal system). On the other hand, the fact that some people may have the faculty (but not the right), the capability and the opportunity to engage in harmful actions towards other individuals creates for the latter also threats, that can be considered liability options, in the sense that they are options in the hands of the de facto counterparts, which, if exercised, will result in liabilities for the people concerned.

Liability Option, Tax Compliance and Evasion

The legal definition of a liability as a "second level obligation" identifies the burden of a party who may be forced to remedy, with an activity or a monetary equivalent, the breach of a rightful relationship. In the framework of a social contract, the case of taxation and the related non-compliance is thus typical in that it arises from a private party violating the "first level obligation" *to give* that existed in favor of the State as its right holder.

Consider the problem of an economic agent who operates in a market where her profits are taxed at a given rate and assume that she has evaded or eluded tax payments for a certain number of years. By undergoing some detection and enforcement costs, the government (or a specific tax agency) can force her to pay a fine plus a tax as a percentage of its revenues for all subsequent periods of production. The costs incurred by the government may include those arising from the bureaucracy, concessions to collection agencies, intelligence, and legal costs. The agent may, in turn, try to remain non-compliant through provisions that entail themselves some private costs in secrecy, deception or legal protection.

The timing of the actions and counter actions is as follows. First, the agent considers the value of the fine for the past evasion and the tax that she would have to pay for all subsequent periods and makes her decision on whether to comply or to continue to evade by undertaking specific costs. These costs are not fixed, but contingent on the level of the agent's expected gains and the corresponding level of the fine and the expected taxation. Although they are not made public, we assume that the tax agency (TA) can estimate them accurately so that it can make enforcing decision based on optimal dynamic planning under uncertainty. The interaction between the firm and the TA is thus assumed to take the form of a dynamic Stackelberg game under uncertainty, with the agent as the leader and the TA as the follower. This implies that, in doing her calculations, the agent, who is assumed to know TA's decision mechanism, takes into account TA's expected future behavior, and optimally sets protection costs. Thus, we first obtain the value of the option to impose sanctions and taxes as a threat for the agent, we then work out the agent's decision and finally we solve the TA's problem.

Let us assume that the net gain from the agent economic activity is governed by a stochastic process of the geometric Brownian motion variety:

(1) $dX_t = \alpha X_t dt + \sigma X_t d\zeta$

where *X* indicates the expected flow of the net gain, *α* and *σ* are, respectively a trend (the "drift") and a volatility parameter, *t* is time and $d\zeta$ is a random variable with zero mean and variance equal to *dt.*

TA, which we assume monopolizes the tax collection activity of the type considered, holds an option to engage in her revenue raising activity, *vis a vis* a given agent, who is suspected to be non compliant, provided it has the evidence and the means to do so. In line with Engelen (2004), in the real option set up, TA has the faculty, but not the obligation, to commit resources to pursue this particular type of activity.

In order to decide whether to exercise her option, once she has met an agent that is suspected of non-compliance, TA solves the problem:

(2) $V_M(X) = \sup_{\tau} E_X \left[e^{-\rho\tau} \left(\int_{\tau}^{\infty} e^{-\rho(s-\tau)} \gamma X_s ds - pC - K \right) \right]$

In words, TA's maximizing problem is given by the net expected discounted value of the future payoff. In (2), the term on the left-hand-side, LHS, represents the value of the option to engage in collecting taxes held by TA. In the right-hand-side, RHS, *ρ* is a discount rate, $\gamma = \theta + s$ is the sum of the tax rate θ, and the sanction rate s, or the share of the agent's income which would be appropriated by TA once it has successfully forced the agent to agree to pay the tax; *p* is a measure of the effectiveness of the protection costs borne by the agent to elude or evade the action by TA, and *K* denotes other costs, including monetary expenditures to perform the tax collection activity, such as hiring employees, collecting information, auditing the agent's accounting, conducting studies on agent's performance, or any other activity necessary to control tax compliance. In accordance with the real options literature, *K* captures also public costs, i.e. the general (non-agent specific) costs that the TA has to commit to proceed against tax evasion. They thus represent an investment whose effects are to deter the agents from evading or eluding taxation. This effect is considered as lasting in the future (the tax collected is a proportion of the present value of expected income of the agent) and so is the effect of the evasion and elusion costs paid by the agent. The parameter *p* can be considered as a measure of differential effectiveness of private costs of non compliance, in terms of public costs (which are taken as numeraire so that they have an effectiveness equal to one). It can also be seen as a conversion factor to translate private costs into social costs. The model is a model of partial equilibrium, in that it only considers the interaction between the TA and the agent, as it is usually the case, with the government and the justice system playing the role of an impersonal machinery that incorporates the social contract.

Proposition 1: TA will commit resources to collect payments from a given subject when the value of her expected gain is greater than or equal to a critical value X_M, at which the value of her option to collect reaches its maximum.

Proof:

as shown in the appendix, the evaluation problem in (2) has a state dependent solution, contingent on whether the value of the stochastic variable *X* is above or below a critical threshold:

(3a) $V_M(X) = \gamma\frac{X}{\delta} - pC - K$ if $X \geq X_M$

(3b) $V_M(X) = (\frac{X}{\delta})^{\beta_1}(\gamma\frac{X}{\delta} - pC - K)$ if $X < X_M$

(3c) $X_M = \frac{\delta\beta_1}{(\beta_1 - 1)\gamma}(pC + K)$

Where $\delta=\rho-\alpha$ is the so called convenience yield, a discount rate measuring the opportunity cost of waiting, β_1 is a parameter inversely related to volatility that can be determined as the positive (and strictly greater than 1) root of the fundamental quadratic expression:

(3d) $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - \rho = 0$

while X_M is the collecting threshold, or entry value. This is the minimum value of *X* that makes TA willing to approach the non-compliant agent and initiate the activity of recovering the taxes that have not been paid. Hence, whenever *X* randomly fluctuating reaches (from below) X_M ,TA will be willing (or even eager) to enter the collection activity. The value of such a threshold increases as the costs to be borne by TA to enter into collection (*pC+K-S*) increase. Comparative statics of (3c) show that an increase in the payoff volatility σ increases the value of the threshold X_M, thereby inducing TA to postpone the exercise of the option, namely the decision to start collecting:

$\frac{dX_M}{d\sigma} = \frac{\partial X_M}{\partial\beta_1}\frac{\partial\beta_1}{\partial\sigma} > 0$, where $\frac{\partial X_M}{\partial\beta_1} < 0$ and $\frac{\partial\beta_1}{\partial\sigma} < 0$ from the fundamental quadratic equation (3d) (see Dixit and Pindyck p. 144).

Condition (3a) states that the highest possible value of the entry option is the expected net present value[2] gained from a successful investment in recovering unpaid taxes. This value is reached

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[2] Assuming dX_t/X_t normally distributed implies X_t lognormally distributed. Given this assumption $E(X_t)=\int_\Omega X_t d\omega = Xe^{\alpha t}$, where *X* denotes the initial value of X_t (i.e. the current value, if the time perspective of the expectation is from the present moment to the future). The present value of this expectation is obtained by discounting it at rate ρ yielding: $\int_\Omega\int_0^\infty X_t e^{-rs} ds d\omega = \int_0^\infty X e^{-(r-\alpha)s} ds = X/(\rho-\alpha)$ with $\rho>\alpha$. See Dixit and Pindyck (1994, p. 71).

for values greater than the threshold, X_M. For values of X lower than the threshold, the option value is below its maximum by a quantity $(X / X_M)^{\beta_1}$, which is a positive function of the ratio between the current and the critical value of X.

From (3c), we can derive the level of elusion and evasion costs that would deter TA from immediate action:

(4) $pC + K \geq G_M$ where $G_M = (\gamma \frac{\beta_1 - 1}{\beta_1} \frac{X}{\delta})$

Inequality (4) represents the maximum value of elusion and evasion costs such that TA does not find it optimal to enter tax recovery activity. Therefore, the agent has the power to prevent TA's action at the expected current value of the gains from entrepreneurial activity, $\frac{X}{\delta}$, by setting private costs at a level that pushes the threshold of TA entry above her expected gains (i.e. at a level such that $X_M>X$). Note that public and private costs are perfect substitutes with a rate of substitution equal to the effectiveness parameter p, which can be interpreted as the unit value of private expenditure for tax avoidance, taking public expenditure against tax avoidance as the *numeraire*. This implies that for any given level of public expenditure K, there is one and only one correspondent level of expenditure on the part of the agent that will achieve the same goal. Hence, for a given value of K, the correspondent value of C is the minimum value of private protection that keeps TA at bay, or, in other words, the maximum value of private non compliance costs that TA can afford to bear.

Given any level of public expenditure, the correspondent level of private evasion costs which can be determined from (4), however, even though sufficient to prevent collection at the current levels of expectations, is not necessarily the best, from the agent's point of view, for two different reasons: (i) first, it may be more expensive to deter TA from collecting than to accept to pay the tax without any evasive action; (ii) second, there may be a level of evasion costs higher than the one indicated by (4) that might convince the TA to desist temporarily from collecting for a wider range of expected gains. Because the firm's gains move in a random environment, in fact, non compliance costs that are at one moment effective to deter TA from acting may be excessive or inadequate at the next moment, due to the fact that expected gains have changed.

Rather than deterring TA from collecting only at the current level of expected gains, therefore, the agent can maintain it in a zone of inaction by increasing the overall level of expenditure for non-compliance by an arbitrary multiple $\mu \geq 1$, provided that this is not more expensive than accepting to pay the tax . More precisely, TA will be confined to a band of inaction given by the interval $(X, \mu X)$ if the agent applies the protection cost:

(4a) $C_\mu = \frac{1}{p}\left[\frac{(\beta_1 - 1)}{\beta_1}\mu\gamma\frac{X}{\delta} - K\right]$

By substituting the value of C_μ given by (4a) into the TA threshold value given by (3c), we find that private cost creates a positive zone of inaction for TA only if $\mu \geq 1$ or, in other words, only if private protection costs are not less than the amount that would deter TA from collecting just at the present level of expected gains.

Corollary: Rather than agreeing to pay the sanction and to become tax-compliant, the agent will prefer to commit resources for non-compliance that raise the TA's threshold of action over current expected gains of a degree of deterrence μ, provided that such a degree is smaller than the sum of relative public and private costs, both adjusted for uncertainty.

Proof:

by comparing the costs undergone by the agent, implied by (4a), to the implicit costs under the compliance alternative $\theta \frac{X}{\delta}$, we find that paying to evade is a more cost effective way than paying the tax if:

(4b) $\frac{\mu}{p}\frac{(\beta_1-1)}{\beta_1}\gamma\frac{X}{\delta}-\frac{K}{p}\leq\theta\frac{X}{\delta}\Rightarrow\left(\frac{X}{\delta K}\right)\left[\mu\frac{(\beta_1-1)}{\beta_1}\gamma-p\theta\right]\leq 1$

Expression (4b) implies that the taxpayer is indifferent between complying and evading if

(4c) $1/\left(\frac{X}{\delta K}\right)=\left[\left(\frac{(\beta_1-1)}{\beta_1}\mu-p\right)\theta+\frac{(\beta_1-1)}{\beta_1}\mu s\right]$

gains from non-compliance increase with income and with the level of the tax (plus sanction) rate. Note that for $\frac{\mu}{p}\frac{\beta_{1-1}}{\beta_1}\leq 1$, the left hand side of the last expression in (4b) becomes negative so that the inequality is always true. If this is not the case and $\frac{\mu}{p}\frac{\beta_1-q}{\beta_1}$>1, on the other hand, expression (4b) implies that the cost from non-complying increases of a multiple of the tax rate so that non-compliance is a preferable alternative only if the tax rate that would be paid is above a threshold defined as in equation (4c) below:

(4c) $\gamma>\left[\left(p-\frac{\beta_1-1}{\beta_1}\mu\right)\frac{X}{\delta K}\right)]^{-1}=\frac{\beta_1}{(p-\mu)\beta_1+\mu}\left(\frac{X}{\delta K}\right)^{-1}$

The threshold level can be interpreted as the maximum tax rate compatible with compliance. It will be positive for $\beta_1>\frac{\mu}{\mu-p}$, i.e. if uncertainty is not too high and will be higher (i.e. it will be more attractive to comply) the lower the income of the candidate tax payer, the lower uncertainty, the lower the cost effectiveness of private elusion and evasion activities p and the higher, coeteris paribus, the size μ of the inaction zone[3]. For example, if $\mu=1, p=1, \beta_1=2$, the threshold will be twice the level of the ratio of taxpayer's income and enforcement cost, while for $\mu=1.5, p=1$ it will be equal to 4 times as much. Thus, if taxpayer's income increases, the maximum tax rate necessary to assure compliance will decrease.

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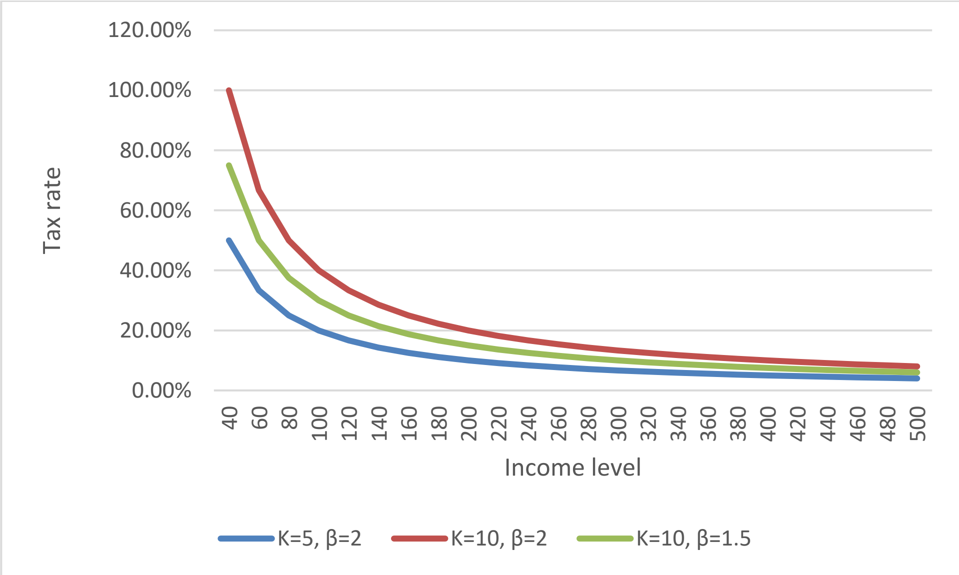
[3] Differentiating the threshold with respect to μ, we obtain: $\frac{\beta_1(\beta_1-1)}{[(p-\mu)\beta_1+\mu]^2}\left(\frac{X}{\delta K}\right)^{-1}$, which is always >0 for $\frac{X}{\delta K}>0$ since $\beta_1>1$ (the variance of the stochastic process is assumed to be finite).

Expression (4b) implies that in the presence of non-compliance, the TA has the option to increase its recovery rate, by reducing the sanction to a level that makes, at a given degree μ/p of the extent of the agent commitment to finance protection from enforcement, non-compliance less attractive for the agent than accepting to pay. The level of the reduced sanction needed to obtain compliance is a negative function of income, since larger incomes make agents less willing to comply, coeteris paribus, to higher tax rates, indicating that a progressive tax schedule (in fact if not in principle) tends to reduce tax compliance, given the same commitment of all agents to incur evasion costs (and risks) as a rational alternative. The parameter μ (the multiple of the minimum cost required to keep TA from enforcing compliance) in (4b) can be interpreted as a degree of the commitment of the agent to avoid enforcement and thus as an index of propensity of persisting in non-compliance on her part. For given values of the propensity to non-compliance μ, private cost effectiveness *p*, and uncertainty parameter β, expression (4b) allows us to draw a diagram showing how the boundary of tax rates over which the agents are not willing to comply is negatively correlated to income levels. As Figure 1 shows, for $\mu = 1, p = 0.25$, the ensuing curve indicates that the maximum tax rate that the agent would be willing to pay without engaging in non-compliance at different levels of public costs *K* and the uncertainty parameter β is decreasing with income. This maximum "self-enforcing rate" divides the space into two subspaces indicating, respectively, the compliance area (above the curve) and the non compliance area (below the curve). The figure shows that the compliance area increases rapidly as the agent's income increases and declines as public costs and uncertainty increase.

Figure 1. Compliance compatible tax rates (Maximum self-enforcing tax rate) and income levels



4. The Stackelberg equilibrium

Consider now the determination of a Stackelberg equilibrium through the commitment of optimal protection costs on the agent's part. For this purpose, we have to consider the objective function of the agent, as an estimate of her contingent wealth, Π_N:

(5) $$\max_C \Pi_N(C) = \frac{X}{\delta} - C - f(V_M(X))$$

The agent maximizes the value of the expected discounted future income, X/δ, net of evasion costs, C, and the value of a function $f(V_M(X))$ depending on the option held by the TA, $V_M(X)$. This function is a contingent liability for the agent, and must be thus considered as a potential source of losses in her objective function.

In order to determine the value of the function $f(V_M(X))$, note that (3b) can be written as:

(5b) $$V_M(X) = Ee^{-\rho\tau}\left(\gamma\frac{X_M}{\delta} - pC - K\right)$$

where $Ee^{-\rho\tau} = \left(\frac{X}{X_M}\right)^{\beta_1}$ is the expected discount factor (the expectation being taken with respect to the time of exercise as a random variable, see Dixit and Pindyck, 1994, pp. 315-316), which is less than or equal to one, according to whether the value of the payoff X is below or above the threshold of entry X_M. Thus, the value of the option for the TA is equal to the expected net present value of her earnings upon and after entry. The value of the threat from the same option for the agent, however, does not include the costs paid by the TA, but only the amount of her earnings that the TA would be able to collect. It follows that for the agent the value of the threat is a function of TA's value function, $f(V_M(X))$, where $V_M(X)$ takes the value found in (3c), and defined as follows:

(6) $f(V_M(X)) = Ee^{-\rho\tau}(\gamma\frac{X_M}{\delta}) = \frac{\gamma}{\delta}\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{pC+K}{\gamma}\right)\right]^{1-\beta_1} X^{\beta_1}$.

Notice that the value of the contingent liability for the agent expressed by equation (6) and corresponding to the TA's option, but with a negative sign, is smaller, the larger, *coeteris paribus*, the effective cost created by the agent private expenditure for TA and the more distant in time, correspondingly, appears the exercise of the collection option by TA. This is simply the result of the fact that greater evasion costs will increase the threshold of entry, enlarge the zone of inaction and reduce TA's option value, thereby improving the agent's expected wealth in (5). However, the value of the agent's liability from the forced collection option decreases less than proportionally with the increase in costs. The optimum value of private protection costs for the agent (the optimal degree of deterrence or the optimum width of the TA inaction zone) will be reached when the decrease in the agent's wealth from one dollar of additional tax evasion costs equals the increase in agent's wealth from the corresponding reduction of one dollar of liability from the TA's option to collect. The optimum *C* is thus the optimum amount of resources spent in tax evasion costs if the agent is aware of the threat but has not come across the government agency yet, i.e. $X<X_M$.

It is possible now to restate the agent's optimal protection problem in (5) as a constrained maximization problem in which the constraint is given by the condition required to create a zone of inaction. Formally, the latter condition is derived by setting $C_\mu>0$ i.e. μ≥1 in (4a). Therefore, the problem in (5) can be rewritten as:

$$\underset{C}{Max}\,\Pi_N(C) = \frac{X}{\delta} - C - f(V_M(X))$$

subject to:

(7) $$\frac{\beta_1}{(\beta_1 - 1)}\frac{(pC+K)}{\gamma\frac{X}{\delta}} \geq 1$$

The solution to this problem is obtained by substituting the definition of *f(.)* in (6) into (5) and solving for *C*:

(8) $$\arg\max_C \Pi_N = C_N = \frac{1}{p}\left[\frac{(\beta_1 - 1)}{\beta_1}\mu\gamma\frac{X}{\delta} - K\right]$$

where $\mu = (\beta_1 p)^{\frac{1}{\beta_1}}$ if $\beta_1 p > 1$, $\mu = 1$ if $\beta_1 p \leq 1$.

Proposition 2. Under the conditions of proposition 1 a preventive Stackelberg equilibrium will exist. If $p\beta_1 > 1$, it will be characterized by a degree of tax evasion where TA will be discouraged from immediate action and left in a zone of inaction, from which it will exit only if the underlying conditions undergo a change larger than $(p\beta_1)^{\frac{1}{\beta_1}}$. Alternatively, if $p\beta_1 \leq 1$ the preventive Stackelberg equilibrium will be characterized by deterrence only at the current value of expected gains.

Proof:

substituting (8) into (3c), we obtain the critical entry point for the TA as in (3c) under the constraint $C = C_N$

(9) $\overline{X}_M = \mu X$,

where $\mu = (\beta_1 p)^{\frac{1}{\beta_1}}$ if $\beta_1 p > 1$, $\mu = 1$ if $\beta_1 p \leq 1$.

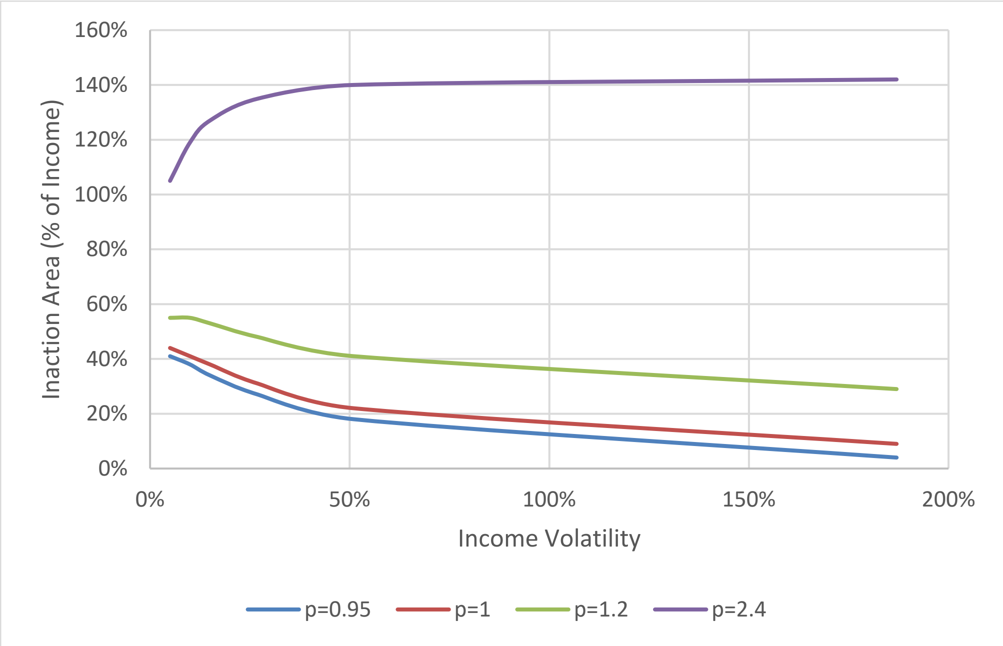
This expression indicates that optimal private expenditure to neutralize tax enforcement prevents the tax collection activity from being exercised, unless the income of the agent rises above the threshold of an amount equal to the factor $(p\beta_1)^{\frac{1}{\beta_1}}$ i.e. X_M>*X*, if $p\beta_1 > 1$. Because the threshold of entry is kept above the expected gain of an amount greater than 1, TA will be in a band of inaction equal to $\overline{X}_M - X = [(p\beta_1)^{\frac{1}{\beta_1}} - 1]X$ and the size of this area as a percentage of current income of the tax payer will equal the difference $p\beta_1 - 1$. Note that this area increases proportionally with the taxpayer income X, so that in percentage terms it has the same value regardless of the income level of the agent, for the same value of cost effectiveness *p*. On the other hand, since higher income agents may have access to more effective ways to elude or evade than lower income ones, the size of the area is likely to be larger the larger the agent's income under high and less uncertain cost effectiveness of private legal and financial expenditure ($p\beta_1 > 1$). Therefore, if $p\beta_1 > 1$, the protection level from tax enforcement in equation (8) will be optimal and effective from the private point of view, and likely to be proportionally larger for larger incomes. If $(\beta_1 p) < 1$ on the other hand, the preventive Stackelberg problem will only have a corner solution equal to the constraint in (7). In both cases, the agent will rationally choose the Stackelberg solution or compliance depending on which of the two alternatives is more convenient.

Corollary. The width of the inaction zone for the TA for $p\beta_1 > 1$ from optimal private evasion costs will be determined by the product $\left((p\beta_1)^{\frac{1}{\beta_1}} - 1\right)X$. This term increases with income and declines or grows with uncertainty depending on whether $p\beta_1$ it is above or below a threshold approximately equal to 2.78 times the income of the agent.

Proof:

differentiating $(\beta_1 p)^{\frac{1}{\beta_1}}$ w.r.t. β_1, we obtain: $\frac{\partial}{\partial \beta_1}(\beta_1 p)^{\frac{1}{\beta_1}} = \frac{1}{\beta_1^2}(\beta_1 p)^{\frac{1}{\beta_1}}(1 - \log(\beta_1 p))$, which is greater than or equal to zero if $1 \geq \log(\beta_1 p)$. Thus, the size of the zone of inaction increases with reduced uncertainty, reaches a maximum at $\log(\beta_1 p) = 1$ i.e. approximately at $\beta_1 p = 2.78$ and then increases with uncertainty.

Figure 2. The size of the inaction zone $(\beta_1 p)^{\frac{1}{\beta_1}} - 1$ for $\rho = 0{,}1$ and $\alpha = 0$ under different levels of cost effectiveness (p) of private expenditure to evade.



Comment: as Fig.2 shows, for moderate degrees of evasion cost effectiveness, the size of the zone of inaction increases with the reduction of uncertainty at a rate that is higher the higher the degree of private effectiveness. *Vice versa*, under sufficiently high cost effectiveness, the larger the uncertainty, the larger the size of the inaction zone. Intuitively, by boosting the cost effectiveness of her legal and financial protections, the agent can take advantage of uncertainty to increase the holding (and reduce the exercising) value of the option to collect and induce the TA to delay its action almost indefinitely.

Corollary. if optimal private expenditure can deter TA from acting (i.e. $p\beta_1 > 1$) the agent will prefer to optimally protect herself and a preventive Stackelberg equilibrium will exist, provided

that public enforcement costs are sufficiently high as compared to the expected loss from taxation at her income level.

Proof:

by comparing the agent's gains from equation (8) to her gains in the case where tax collection occurs, and using (6), we find that the agent's gain under optimal private expenditure is:

(9a) $\Pi(C_N) = \frac{X}{\delta} - \frac{1}{p}\left(\mu\gamma\frac{X}{\delta} - K\right)$

where $\mu = (\beta_1 p)^{\frac{1}{\beta_1}}$ if $\beta_1 p > 1$, $\mu = 1$ if $\beta_1 p \le 1$.

Under no private expenditure the agent expected gain would be:

(9b) $\Pi(C = 0) = \frac{X}{\delta} - \frac{\gamma X}{\delta}\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{K}{\gamma X}\right)\right]^{1-\beta_1}$ for $\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{K}{\gamma X}\right) > 1$

= $(1-\gamma)\frac{X}{\delta}$ for $\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{K}{\gamma X}\right) \le 1$

Note that the term in square brackets in (9b) is the ratio between the threshold of action of TA with no private expenditure and the expected size of tax collection. If *K/γX* is sufficiently large such that this term is greater than 1, (which will happen the higher the enforcement costs, and the lower the income and the tax rate), the agent will have no need to protect herself from TA since public enforcement costs are high enough to deter TA from action at the current level of her expected income. If the term in brackets is lower than or equal to 1, on the other hand, TA can be expected to act. By comparing (9a) and (9b), under the latter hypothesis, we obtain that $\Pi(C_N) > \Pi(C = 0)$, if the following condition holds

(10) $\frac{\beta_1 - 1}{\beta_1}\frac{\gamma X}{\delta} \ge K \ge (\frac{\mu}{p}\frac{\beta_1 - 1}{\beta_1})\gamma - (1-\gamma)]\frac{X}{\delta}$ if $\mu = (\beta_1 p)^{\frac{1}{\beta_1}}$ i.e. $\beta_1 p > 1$,

$(\mu - p)\frac{\gamma X}{\delta} \le K \le \frac{\beta_1 - 1}{\beta_1}\frac{\gamma X}{\delta}$ if $\mu = 1$ i.e. $\beta_1 p \le 1$.

Thus, non-compliance with private evasion costs will be preferred to tax compliance and a preventive Stackelberg equilibrium will be reached only if public collection costs are sufficiently high in relation to the size of the tax collection, the income of the agent and the uncertainty and the degree of private effectiveness.

Under several alternative circumstances, therefore, TA will be deterred from immediate action and a (temporary) preventive equilibrium will be reached, because the agent income is uncertain, enforcement costs are high and so is the related tax that can be collected. Note again that the higher the income of the agent, the higher, coeteris paribus, compliance for a given tax level. Under the conditions described, in fact, higher incomes imply higher non-compliance costs and therefore, lower degrees of non-compliance, for a rational agent. Higher tax rates (and or sanctions), for any income level, can also reduce non-compliance, but only if the ensuing tax collection can be achieved without increasing, at the same time, public enforcement costs, as well as uncertainty. On the other hand, as demonstrated by the size of the inaction zone found above, higher income non-compliant agents are also associated with comparatively wider ranges of the inaction area for the TA.

Proposition 3. Equilibrium between the agent and TA will be characterized by two regimes, depending respectively, on whether the agent chooses to protect herself or to pay the tax. Which regime will prevail will depend on whether the ratio between the expected value of the tax to be collected and public enforcement costs is large enough to induce TA to invest in enforcement under no private evasion costs, and small enough so that that private costs are not higher than the taxes to be paid.

Proof:

expression (8) implies that the optimal private cost level will be nonnegative (i.e. the agent will costly non-comply) only if the tax level is sufficiently high, and/or public enforcement costs are sufficiently low. On the other hand, expression (10) shows that if public costs are not sufficiently high, the agent will prefer to comply rather than costly evade. This result can be summarized by means of the following double inequality:

(11) $\frac{1}{\mu}\frac{\beta_1}{\beta_1-1} < \frac{X}{\delta K} \leq \frac{1}{\mu-p}$ if $\mu = (\beta_1 p)^{\frac{1}{\beta_1}}$ i.e. $\beta_1 p > 1$

$\frac{X}{\delta K} = \frac{\beta_1}{\beta_1-1}$ if $\mu = 1$ i.e. $\beta_1 p \leq 1$.

We thus have two possible regimes corresponding to whether the ratio between the agent' income and public protection costs falls within or outside the interval defined by (11). In the first regime, where public protection is within the boundaries defined by the inequality, a preventive Stackelberg equilibrium exists and the agent actively protects herself, thereby preventing any enforcement move on the part of the TA. In the second regime, instead, the agent chooses not to protect herself either because public protection is so high, and or, the agent's income is so low that optimal deterrence is already achieved (the second inequality of (11) does not hold), or because public protection is so low that the private costs required make tax compliance a better deal than costly evasion (the first inequality of (11) does not hold). In this case, a preventive Stackelberg equilibrium does not exist. Changes in the variables of equation (7) can thus have either incremental or pivotal effects, according to whether these do not determine or determine a regime switch, respectively.

Corollary. Changes in public protection, private protection effectiveness, uncertainty and expected gains will have different effects on the tax collection level according to whether they are incremental (they do not cause a change in regime) or pivotal (they cause a change in regime).

Consider the ratio between the agent's income and public protection costs. The lower this ratio, the higher the degree of deterrence (the wider the zone of inaction) that the agent can achieve for TA by adding private protection. Thus, if this ratio is small enough, the agent will be motivated to evade and protect herself with legal and financial expenses for the time being. Under these conditions, an increase in income may induce the agent to reduce private protection if it is of the incremental type. An increase in income, however, may have the opposite effect if it is pivotal, i.e. it brings the agent above the threshold that makes costly evasion less attractive than tax compliance. Conversely, if we are in the opposite regime, with the ratio between income and public protection above the upper bound threshold of equation (11) and no private protection, an income reduction will have no effect unless it is large enough to motivate the agent to switch from compliance to costly evasion.

Looking more closely at the effects of private protection effectiveness, we also see that a change in *p* can have two opposite effects. For the optimal protection regime $(\beta_1 p > 1)$, an increase (decrease) in *p* merely increases (decreases) TA's (forced) band of inaction. A sufficiently large reduction of *p* may be pivotal if it causes a change in regime and pushes the agent in the zone where it is more convenient to risk enforcement. On the other hand, if $(\beta_1 p \leq 1)$, an increase in private protection effectiveness has no effect on tax collection if it is merely incremental. Only if it is sufficiently large to determine a pivotal change (i.e. a regime change) so that the agent can act to confine the TA in a sufficiently large zone of inaction, it will reduce the expected level of tax collection.

2. The tax rate as a social contract

If taxation is seen as one of the stipulations of a social contract, a Nash solution to the conflict of interest between the TA and the tax payer may be sought in the context of the theory of cooperative games. This means that rather than setting her private protection costs at the optimal level, or comply with the tax payment, the agent may negotiate with the TA to find a mutually acceptable rate of taxation.

Proposition 4. The Nash equilibrium schedule of tax collection from a bargaining game between the tax paying agent and the tax authority is progressive in agents' incomes, with the degree of progressivity increasing with TA's bargaining power, and decreasing with public collection costs.

Proof:

in order to specify the objective functions of the two parties, assume that the social contract is stipulated by starting from a common proportional level of taxation γ_0 and consider that, depending on private protection effectiveness and uncertainty, an agent may be in two distinct regimes: one where the alternative is to protect herself optimally (equation (11)), and one with an incentive to negotiate with the TA through legal or political processes .

Consider the regime where the agent does not protect herself optimally because such a protection would be more costly than complying. In this case, the agent's present wealth is:

(12) $\Pi_A = (1-\gamma_0)\frac{X}{\delta}$

where γ_0 is the tax rate unilaterally fixed by the tax authority. Alternatively, the agent may choose to wait for the time TA exercises its option. In this case her pay off present value is given by:

(13) $\Pi_A = \frac{X}{\delta} - \frac{{\gamma_0}^{\beta_1}}{\delta}(\frac{\delta\beta_1}{\beta_1-1})^{1-\beta_1}X^{\beta_1}$,

where γ_0 is the tax rate that the agent expects to pay at the time TA exercises her option.

I assume that, if the agent agrees to pay, TA will have no costs, so that its expected value is simply: $\Pi_T = \gamma_0\frac{X}{\delta}$, while, if the agent does not agree, TA's expected gain $\Pi_{T,A}$, is the value of her option to enforce the tax, which in this particular case, can be obtained by substituting (3c) into (5b) and by setting *C=0*:

(14) $\Pi_{T,A} = (\frac{X}{X_M})^{\beta_1}(\gamma_0 \frac{X_M}{\delta} - K) = \frac{\gamma_0{}^{\beta_1}}{\delta}(\frac{\delta\beta_1}{\beta_1 - 1}K)^{1-\beta_1}$

Notice that X_M comes from (3c) rather than (9) because the latter formula is consistent with the threshold X_M evaluated at $C=C_N$, which clearly does not apply in the case we are analysing.

Hence, in this case the Nash objective function can be specified as follows (Nash, 1951; Harsanyi, 1967, 1968):

(15) $N(w) = (\Pi_B - \Pi_0)^w (\Pi_{OC} - \Pi_{OC,B})^{1-w} = \left[\frac{\gamma_0}{\delta}\left(\frac{\delta\beta_1}{\beta_1 - 1}\frac{K}{\gamma_0}\right)^{1-\beta_1} X^{\beta_1} - \gamma\frac{X}{\delta} \right]^w \left[\frac{\gamma X}{\delta} - \frac{\gamma_0}{\delta\beta_1}\left(\frac{\delta\beta_1}{\beta_1 - 1}\frac{K}{\gamma_0}\right)^{1-\beta_1} X^{\beta_1} \right]^{1-w}$

where $N(w)$ denotes the Nash objective function and $0<w<1$ is a weight measuring the bargaining power of the agent. This weight is plausibly increasing in the agent's income i.e. $w = w(X)$, $w' = \frac{\partial w}{\partial X} \geq 0$. Differentiating with respect to γ, equating to zero and solving for γ, we obtain[4]:

(16) $\gamma^* = \arg\max N(w) = \gamma_0^{\beta_1}\left(\frac{\beta_1}{\beta_1 - 1}\frac{K}{X/\delta}\right)^{1-\beta_1}\left(1 - w\frac{\beta_1 - 1}{\beta_1}\right)$

Taking the derivative of (16) w.r.t. X/δ, we obtain:

(17) $\frac{\partial\gamma^*}{\partial(\frac{X}{\delta}/K)} = \gamma_0^{\beta_1}(\frac{\beta_1}{\beta_1 - 1}\frac{K}{\frac{X}{\delta}})^{-\beta_1}\frac{1}{\frac{X}{\delta}/K}[\frac{1}{(\frac{X}{\delta}/K)} - w']$

Expression (16) shows that, for a given distribution of bargaining power, the Nash equilibrium level of tax collection is higher the higher the agent's income, and the lower public protection (recall that $1 - \beta_1 < 0$). Thus, if the bargaining power of the tax payers were uniform, a progressive tax schedule would be a superior choice from the point of view of effectiveness of tax collection, as well as of the balance between private and public interests. However, given that the bargaining power and incomes are likely to be positively correlated, expression (17) indicates that we have to expect that the ultimate shape of the tax schedule across agents will depend the balance between the public interest (higher tax collection effectiveness measured by $(K/\frac{X}{\delta})$ and the political influence of higher income agents ($w' = \frac{\partial w}{\partial X}$)). Public protection also appears to have ambiguous effects on the rate of taxation. On the one hand, as demonstrated by inequality (11), if it is sufficiently large, it will encourage the agent to protect herself optimally. On the other hand, if it is not sufficient to motivate the agent to seek a Stackelberg equilibrium, as shown by (16), by reducing

4 $d^2N(w)/d\gamma^2 < 0$ so that γ^* is a maximum.

the Nash level of agreeable taxation, public protection may encourage the firm to enter willingly in a social contract.

6. Conclusions

In this paper, by relying on the real options framework, I have modelled the relationship between a private agent and a tax authority, under dynamic uncertainty. The relationship between the tax authority (TA) and the private agent is a special case of an implicit real option exchange, where the TA holds a real option to enforce tax collection, of the call type, at the expenses of the agent. In finding herself to hold (to be long in) such an option, against a fix cost to exercise it, the TA faces the prospect of gaining an uncertain upside linked to the tax due by the agent. On her part, the agent is in a condition equivalent to be short in the same option, as if it had originated from a sale of her position to the tax collector. This "liability" option thus represents a threat for the non-compliant tax payer, since it can generate an uncertain downside depending on the action that the TA may decide to take.

A remarkable result of the development of this model, is the suggestion that a schedule of progressive taxation may be entirely justified by cost effective considerations. Higher income tax payers in fact would be threatened by proportionally larger liability options, whose size would also increase with uncertainty, and would thus be more motivated to comply the larger is, coeteris paribus, the tax level (including the sanction) that they would be required to pay in the case of non-compliance. At the same time, if we assume, as it seems reasonable, that higher income agents are likely to have more influence on the political process, the model predicts that this result may be reversed and the most likely equilibrium solution emerging from a Nash negotiation between the tax authority and the agents would lead to a proportional or even a regressive system.

The real option model appears also to carry several implications that contradict the conventional wisdom of the crime and punishment as well as of the classical tax enforcement literature. First, while the relative burden of the liability option, as measured by the cost that the agent would have to undertake to neutralize enforcement, would be larger the higher her income, similarly larger would be the correspondent optimal size of the inaction zone where the TA will be confined by the legal and financial action of the non-compliant tax payer. Second, and symmetrically, as outlined as a major conclusion above, the maximum tax rate that the agent would be willing to pay without engaging in non-compliance at different levels of public costs and uncertainty is decreasing with income, implying that a progressive tax schedule is more cost effective for the tax collector. Third, under a Stackelberg game (see Shubik, 1991, pp. 85-86) with the agent as the leader and TA as the follower, depending on the taxation level, uncertainty and the agent's expected payoff, three alternative conditions may prevail: (i) one in which the non-compliant agent protects herself optimally from TA enforcement, with an ensuing Stackelberg equilibrium in which TA is in a zone of inaction, that is larger, the larger are uncertainty and private protection effectiveness , (ii) one in which the agent will **not need** to protect herself because of public inefficiency (high public costs of enforcement) and, (iii) one in which the agent will **prefer** to comply because public enforcement costs and/or private protection effectiveness are so low (uncertainty is so high) that compliance is more convenient than elusion or evasion.

Fourth, a Nash equilibrium may be seen as the result of an explicit or implicit negotiation of a social contract between the tax authority and each different tax payer. In this case, two Nash bargaining solutions in general exist, depending on whether the relevant agent's alternative at the initial tax level is compliance or evasion. Both Nash solutions will imply compliance and will be characterized by a level of taxation depending on the balance between the public interest (higher tax collection effectiveness) and the political influence of higher income agents.

Fifth, the higher the uncertainty, the lower should be public enforcement in order to discourage the agent from pursuing an optimal protection policy under non-compliance. As a consequence, in order to foster economic activity in regions suffering from high levels of the shadow economy, the government can directly operate through an increase in the effectiveness of public enforcement (more efficient judicial and enforcement system, certainty of punishment, lower times of court actions, etc.) and reduce the intrinsic risk of economic activity, by providing opportunities for insurance and diversification.

Our results also suggest that private and public protection activities have to be carefully balanced since they act as imperfect substitutes of one another in favoring or fighting tax evasion, for five related reasons. First, higher public protection is non-specific and is equivalent to a general increase in the investment costs that TA has to bear to engage in enforcing tax collection. As such, it will tend to reduce the private costs that the private agent has to bear, *coeteris paribus*, to achieve the same level of deterrence of TA from enforcement. Second, lower costs and/or higher cost effectiveness of the public authority, by making more attractive the alternative of dealing or bargaining with the TA, may reduce the incentive for the private party at risk to engage in evasive activities and encourage the formation of trust and confidence between the public and the private sector. This effect may be especially dramatic in the case of a "pivotal change", that is in the case that increasing or decreasing public effectiveness causes the agent to emerge from or to plunge into the shadow economy. While a sufficiently high public effectiveness may eliminate any incentive for the agent to join the ranks of the tax evaders, the failure of public cost effectiveness to reach a critical amount may be detrimental since it may induce the agent to negotiate with the TA and result in an increase in the level of compliance and legality of the system. Third, changes in combined private protection effectiveness and uncertainty will have effect on crime deterrence only in the regime of optimal private protection. At the same time, their joint consequences will be decisive only if they are "pivotal" in the sense that they lead to a regime switch.

APPENDIX

1. **Proof of Proposition 1.**

In solving the problem in (2) I have adopted the approach of Dixit and Pindyck (1994, pp.122-123). First, expressions for the value of the investment in the continuation and stopping regions are derived by solving the relevant Bellman equations. In the continuation region TA holds an option to invest, analogous to a call option, while in the stopping region the irreversibility of investment entails that its value is simply the expected value of the project. Next the boundary between these regions, the optimal stopping point X_M, is found by imposing optimality conditions, value-matching and smooth-pasting conditions. The statement that TA invests at the stopping point X_M means that it invests at the time when the stochastic process X first crosses the value X_M, approaching this level from below.

Dynamic programming requires that the following no-arbitrage condition holds:

(A1) $E[dV(X)] = \rho V(X)$

By applying Ito's lemma and taking the expected value we can write the following Bellman equation:

(A2) $\alpha V' X + \frac{\sigma^2}{2} V'' X^2 = \rho V$

whose general solution is of the type

(A3) $V_M(X) = A_1 X^{\beta_1} + A_2 X^{\beta_2}$

Where β_1 and β_2 are respectively the positive and negative root of the characteristic equation

(A4) $\frac{1}{2}\sigma^2 \beta(\beta - 1) + \alpha\beta - \rho = 0$

and A_1 and A_2 are two constants to be determined from the boundary conditions. Since the value of the investment option increases as the underlying increases, A_2 must equal zero and $V_M(X)$ becomes

(A5) $V_M(X) = A_1 X^{\beta_1}$

In the stopping region, TA optimally exercises the option gaining the expected value of the investment, therefore the value function is:

(A6) $V_M(X) = \gamma \frac{X}{\delta} - pC - K$ for $X \geq X_M$

In the continuation region, TA waits to exercise the option and the value function is worth the discounted expected value of future gains,

(A7) $V_M(X) = E_t(e^{-\delta\tau}) A_1 X^{\beta} = E_t(e^{-\delta\tau}) \left(\gamma \frac{X_M}{\delta} - pC - K \right)$ for $X < X_M$

as in (3b) in the text.

The value-matching condition matches the values of the unknown function, $V_M(X)$, to those of the known termination payoff function, namely the expected value of the investment. Therefore, the value-matching condition can be written as:

(A8) $A_1 X^{\beta} = \gamma \frac{X}{\delta} - pC - K$

The smooth-pasting condition requires the value functions to meet at the stopping point with equal first derivatives and can be obtained by deriving the value-matching:

(A9) $A_1 \beta X^{\beta - 1} = \frac{\gamma}{\delta}$

By combining the two conditions and solving for *X*, we obtain the optimal stopping point as in (3c) in the text.

(A10) $X_M = \frac{\delta \beta_1}{(\beta_1 - 1)\gamma}(pC + K)$

It is also possible to prove (see Dixit and Pindyck 1994, p.315-316) that the discount factor $e^{-\delta\tau}$ can be rewritten as $\left(\frac{X}{X_M} \right)^{\beta_1}$ from which we find (3b) in the text.

2. Proof of Proposition 2.

Consider the problem:

(B1) $\underset{C}{Max} \Pi_N(C) = \frac{X}{\delta} - C - f(V_M(X))$

(B2) subject to: $\frac{\beta_1}{(\beta_1 - 1)} \frac{(pC + K)}{\gamma \frac{X}{\delta}} \geq 1$, where

(B3) $f(V_M(X)) = \frac{\gamma}{\delta}\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{pC+K}{\gamma}\right)\right]^{1-\beta_1} X^{\beta_1}$

Forming the Lagrangian, we obtain:

(B4) $L = \frac{X}{\delta} - C - f(V_M(X) - \lambda\left[\frac{\beta_1}{(\beta_1 - 1)}\frac{(pC+K)}{\gamma X/\delta} - 1\right]$,

λ being a Lagrange multiplier.

The Karush- Kuhn-Tuker conditions (Karush, 1939: Kuhn -Tucker, 1951) require that

$\frac{\partial L}{\partial C} \leq 0$ C≥0 and $C\frac{\partial L}{\partial C} = 0$

$\frac{\partial L}{\partial \lambda} \leq 0$ $\lambda \leq 0$ and $\lambda\frac{\partial L}{\partial \lambda} = 0$

Therefore, taking the derivatives we have:

(B5a) $\frac{\partial L}{\partial C} = -1 + p\beta_1\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{pC+K}{\gamma}\right)\right]^{-\beta_1} X^{\beta_1} = 0$ and $\lambda = 0$

or

(B5b) $\frac{\partial L}{\partial C} = -1 + p\beta_1\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{pC+K}{\gamma}\right)\right]^{-\beta_1} X^{\beta_1} - \lambda\left(\frac{\beta_1}{(\beta_1 - 1)}\frac{p}{\gamma X/\delta}\right) = 0$ and $\lambda < 0$

The constraint is not binding, i.e. $\lambda = 0$, if $\beta_1 p$ >1 and we have from (B5b) as a solution.

(B6a) $C^* = \frac{1}{p}\left[\frac{(\beta_1 - 1)}{\beta_1}(\beta_1 p)^{\frac{1}{\beta_1}}\gamma\frac{X}{\delta} - K\right]$

On the other hand, if the constraint is binding ($\beta_1 p \leq 1)$, the solution is simply the value of private cost consistent with the constraint as an equality:

(B6b) $C^* = \frac{1}{p}\left(\frac{\beta_1 - 1}{\beta_1}\gamma\frac{X}{\delta} - K\right)$

In this case, the maximum reduction of the threat that private protection can achieve coincides with the level of protection that prevents TA from trying to extort at the current level of expected gains. Any level of protection below (B6b), in fact, would have as a consequence the immediate action on the part of the TA.

The value of λ^* can be obtained by substituting the value of *C** given by (B6b) into (B5b) and solving for λ^*

(B7) $\lambda^* = \frac{\beta_1 - 1}{\beta_1 p}\gamma \frac{X}{\delta}(\beta_1 p - 1) \quad \beta_1 p \leq 1$

Recall that the value of the multiplier at the constrained optimum is negative because the constraint binds from below.

To prove that (B6a) is a maximum for $\lambda = 0$, we differentiate again (B5a) w.r.t. the private cost C:

(B7) $\frac{\partial^2 L}{\partial C^2} = -\frac{(p\beta_1)^2}{\gamma}\frac{\delta\beta_1}{\beta_1 - 1}\left[\frac{\delta\beta_1}{\beta_1 - 1}\left(\frac{pC + K}{\gamma}\right)\right]^{-\beta_1} X^{\beta_1}$

This expression is always negative for *C, K>0.*

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